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Designing an Optimal Control Model of a Capacity, Multi-stage Continuous MRP System Considering Delay in Production Lead Time and the Possibility of Reworking and Recycling

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A B S T R A C T

In this paper, we propose a multi-stage continuous-MRP system using an optimal control model, considering the production lead time. In the proposed model, the lead time is specified for ordering work in process during the second stage and for final product manufacturing. Also, the intended dynamical system is a multi-stage production-inventory system that follows a linear-quadratic optimal control model with a time delay for the state variables. In the proposed system, inventories are considered state variables, and the delivery levels and orders are control variables. The return stage is considered for items whose production has been defective. According to their situation, there are three destinations: the reworking stage, the recycling stage, or disposal. The amount of shipment to the next stage of production is based on their BOM utilization coefficient and the inventory one. This stage will consume all sent items at any time and will not create a surplus inventory. In this paper, time is considered a continuous parameter proportional to the constant production processes. For validation, the proposed optimal control model was simulated in a real study in the polymer industry.

Keywords

Optimal control, Continuous material requirements planning (CMRP), Deteriorating items , Lead Time, Reworking, Recycling.

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1. Introduction

Maintaining an inventory of raw materials incurs significant costs for manufacturing companies, particularly in the mass production industry, prompting managers to consider more carefully how to find the optimal solution to this problem. The primary achievement of a required material planning system is the planning of part and goods production, as well as the supply of raw materials, to ensure the production line can meet the required quantities at the necessary time. Proper scheduling of raw materials purchases, on the one hand, and timely ordering of parts for production, on the other, will lead to more reliable access to the raw materials needed on the production line and the timely delivery of products according to a comprehensive production schedule. If material requirements planning is successfully implemented, overall inventory costs will be significantly reduced. Most studies on finite- or infinite-capacity MRP systems focus on discrete-time systems. However, precise material requirements planning is required at a consistent time. Discrete materials requirements planning (DMRP) approach defines orders, demands, and products at a discrete time or Specific periods. Therefore, the discreteness in the DMRP approach implies a breakdown in production time. In this approach, the production parameters are defined over a fixed time. Thus, the available inventory is determined from the beginning of the period.

In contrast, in Continuous Material Requirement Planning (CMRP) systems, the production parameters are determined at each moment of the planning time horizon. In some industries, such systems need to be considered because of their continuous production. In the petrochemical and petroleum industries, for example, production occurs continuously, and inventory levels are constantly changing. However, most of the research become established. On the other hand, in the last decade, the use of optimal control theory in production planning and inventory control has increased. Therefore, the literature on MRP and the application of optimal control theory in production inventory systems are discussed for better understanding.

Sadeghian (2011) first introduced the CMRP approach and subsequently discussed its advantages over the DMRP. He then presented a three-step algorithmic method for determining scheduled orders in discrete MRP. In this algorithm, the scheduled orders were calculated without an optimization approach. Louly et al. (2012) attempted to plan components in assembly systems using the MRP approach. In their proposed model, the lead time was randomly selected. Grubbstrom and Tang (2012) examined the effect of demand scheduling and BOM conditions on the Solution space for production times. Milne et al. (2015) developed a mixed-integer programming approach to determine the optimal lead time for use in MRP

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systems for discrete component production and a fixed-period batch size policy. Rossi et al. (2017) suggested a capacity compatible with the MRP model. In their proposed model, finite capacity was taken into account. They also determined a fixed lead time.

Le Thi and Tran (2014) discussed a plan to optimize the costs of setting up, transporting, and inventing a multistage production system. Their optimization model was a nonlinear, complex integer programming model. Mezghiche et al. (2015) considered an integrated production forecasting system of tracking type. In their proposed system, the demand rate over a given period depends on the demand rate of the previous period and the stock level. Hatami-Marbini et al. (2020) considered a network of manufacturing machines based on the hedging point policy where the final goods were perishable, and the demand rate was constant. Lee et al. (2021) proposed a double-ended queuing model having backorders and customer abandonment. One side of their model stores back orders, and the other side represents inventory.

Foul et al. (2012) presented an adaptive inventory system with an unknown deterioration rate. Vercraene and Gayon (2013) focused on the optimal control model of a production inventory system with disposable items and product returns. Pooya and Pakdaman (2021) presented a manpower planning problem from a continuous-time optimal control. Dhaiban (2022) proposed a new approach to finding integer solutions for the product inventory control model under a periodic review policy. Their approach was based on the modified equations of the Pontryagin maximum principle. Singer and Khmelnitsky (2021) considered a stochastic production-inventory problem and solved it using optimal control methods. ElHafsi et al. (2021) addressed the problems faced by manufacturers in dealing with customers with diverse needs and determining which customers to serve when supply is limited, within a continuous-time integrated manufacturing and inventory control framework. Gao (2022) extended the production-inventory model with a stochastic term, and instead of analyzing steady states or guessing the shape of the solution, they focused on PDEs. She applied the operator division method after the Cole-Hopf transformation to the initial equation, rather than solving the ODE system. Shen et al. (2022) investigated the uncertain production-inventory problem with deteriorating items and developed an optimal control model in their research. They applied uncertainty theory to derive the optimal equation. Nakhaeinejad et al. (2023) used the optimal control theory to propose an order system and inventory management model. In their model, the order was regarded as a time-dependent function and a control variable. In addition, the need for each item was time-dependent and specified. The order and inventory system was indicated as an optimal control problem.

Hedjar et al. (2012) employed optimal control theory to determine the optimal production rates in a production system. Mezghiche et al. (2015) presented a unified production forecasting system using the control approach. In their proposed system, the demand rate over a given period depends on the demand rate of the previous period, and the stock level. Benkherouf et al. (2016) proposed an optimal control model for production and reworking in the inventory system with finite planning horizons. Dizbin and Tan (2020) considered the production control problem of a production-inventory system with correlated demand, inter-arrival, and processing times. They proved that the optimal control policy is the state-dependent threshold policy. Al-Khazraji et al. (2021) scrutinized the dynamic performance of the APIOBPCS model and the newly modified 2APIOBPCS model for optimal control of production–inventory systems in their study. Öztürk (2021) investigated an economic production quantity model that explored the impacts of a flawed manufacturing process with stochastic breakdowns and inspection policies on a manufacturing inventory system with a repair, and determined the optimal production run time.

Pooya and Pakdaman (2017) proposed an optimal control model for production-inventory models. They solved the proposed optimal control problem with a neural network approach. In most models discussed in the production field, it is assumed that production occurs in a single stage. They also (2018) proposed a delayed optimal control model for a multi-stage production-inventory system with production lead times. Still, their planning approach was not MRP and merely managed the inventory of the various stages. In another article (2019), they proposed an optimal control model for finite capacity continuous MRP, which does not consider production lead time, including returned, reworked, and recycled goods with deteriorating items. Rachih et al. (2022) addressed a discrete inventory control model for a reverse logistics system in their paper. In their work, the optimal control policy consisted of minimal levels for stocks of returns and new products and higher levels for remanufactured products. In contrast, the demand for new products was higher than the demand for remanufactured products. Ignaciuk (2022) examined the perspectives of linear-quadratic (LQ) optimal control in steering the process of goods distribution in logistic systems with multiple transportation options. Megoze et al., (2022) presented a study on a manufacturing and remanufacturing system that degrades according to its production rate. After formulating the optimal control problem that described the studied system, they solved the HJB equations using a numerical method.

In many cases, production phasing and planning for each stage separately may be more desirable. Primarily, when using equipment and machinery in separate, sequential applications, it is very cost-effective because it allows for flexibility in planning and reduces production costs with a more accurate schedule. However, multi-stage production systems still occur despite the production lead time, which neglect leads to incorrect planning and either increasing or decreasing inventory. On the other hand, considering that in such production systems, there is a possibility of defective goods after each stage, it is necessary to consider the return stage to rework the returned product or recycle it into the raw materials of the previous step.

Table 1 provides a summary of the research conducted in the field of the present study. The earlier models each had deficiencies in the multi-stage continuous MRP system, considering the finite capacity, the delay in production lead time, and the possibility of re-working and recycling. Many of the previous papers have proposed discrete-time models.

Grubbstrom et al. (2010) considered no capacity constraints. Ignaciuk and Bartoszewicz (2010) dismiss the disposal phase of deteriorating items. Sadeghian (2011) presented a continuous MRP model paper with no optimization approach. Louly et al. (2012) regarded lead time as a random variable. Vercraene and Gayon (2013) presented a two-stage model that ignored recycling. The proposed models did not apply to production systems, despite the industries' production nature being continuous, such as the petroleum and petrochemical industries, as well as some of the products in the construction industry. Benkherouf et al. (2016) presented a model that lacked a recycling stage. Pooya and Pakdaman (2018) did not consider the process of disposing of deteriorating items and reworking. In another paper (2019), they presented a model in which production LTs were considered as zero. Al-Khazraji et al. (2021) presented a continuous optimal control model that considered production lead time but did not account for the return, rework, and recycling stages. Öztürk (2021) proposed a delayed optimal control model with a reworking stage, but in the proposed model, the stages of recycling and disposal of perishable items were neglected. ElHafsi et al. (2021) and Gao (2022) proposed continuous models that do not account for production lead time. In their models, they dismiss the stage of disposal of deteriorating items. Shen et al. (2022) and Megoze et al., (2022) investigated the production-inventory problem with deteriorating items, where the production LTs were considered zero. Rachih et al. (2022) addressed a discrete inventory control model in their paper. Although they included the rework stage in the model, the production lead time was ignored. The challenge observed in earlier models is the lack of a comprehensive model to be proposed for a multi-stage continuous MRP system with finite capacity, considering production LTs with deteriorating items.

Yan and Sun (2024) employ constrained reinforcement learning to solve an optimal control problem aimed at reducing fuel consumption in hybrid energy systems. By accounting for uncertainty and fluctuations in input data (such as sensor errors or changes in environmental conditions) and integrating prediction with dynamic decision-making, they propose a multi-stage framework that can be adapted to MRP systems with rework and recycling capabilities. Luo et al. (2024) focus on a model predictive control (MPC) approach to optimize computations in the presence of input delays. By predicting future states and solving the optimal control problem one step ahead, they significantly reduce delays in implementing controls.

In this study, we introduce an optimal control multi-stage continuous MRP system with finite capacity, considering production LTs with deteriorating items. Purchase and manufacturing orders, shipments, and quantities returned from the market were considered control variables, while inventories were treated as system states. Production LTs are to be the time required for the production and purchase orders. In the proposed multi-stage system, it is permissible to move the returned items from one inventory to the previous one due to the recycling and reworking stages. An essential advantage of the proposed model, which was not addressed in previous models, is that the number of items sent to the next stage must be a coefficient of their use in the BOM as well as a coefficient of inventory. In other words, the number of goods shipped at any given moment in time should be large enough to be used in production at all times and not lead to a surplus inventory of any item at a later stage. The bottom line of Table 1 shows the features of the present study. There are two main steps to be solved by the theory of optimal control: the modelling stage, or model description, and the model solving stage.

The first section of this study is (Model Descriptions) devoted to modelling and model description. In this section, the proposed model and related diagrams are presented. Also, two mathematical models for the delay and non-delay modes are offered. In the second section (Model solving), the proposed linear-quadratic optimal control model is analyzed and solved. In the third section (numerical simulation), the proposed model is solved using the actual parameters studied. The fourth section (Sensitivity Analysis for Q, K and R) is devoted to evaluating the proposed model. Finally, the last section (Discussion and Conclusion) includes a discussion of model features, conclusions, and future research suggestions.

Table 1. Previous research on MRP mathematical models and challenges

Author	Topic	Type of Model	Model Challenge: Discrete Time	No possibility of reworking	One-stage	Without LT	Infinite capacity	No possibility of recycling	No optimization	BOM is not allowed
Sadeghian (2011)	MRP	MRP							*	
Grubbstrom et al. (2010)	MRP	MRP	*				*			
Rossi et al. (2017)	MRP	MRP	*							
Milne et al. (2015)	MRP	MRP	*							
Louly et al. (2012)	MRP	MRP	*							
Grubbstrom and Tang (2012)	MRP	MRP	*							
Le Thi and Tran (2014)	MRP	Integer nonlinear programming model	*							
Hedjar et al. (2012)	Optimal control applications in production planning and inventory control	Continuous optimal control						*		
Dizbin and Tan (2020)		Continuous optimal control		*				*	*	*
Mezghiche et al. (2015)	"	Continuous optimal control		*				*		
Pooya and Pakdaman (2017)	"	Continuous optimal control		*		*		*		*
Pooya and Pakdaman (2018)	"	Continuous optimal control		*				*		
Pooya et al. (2019)	"	Continuous optimal control			*					*
Pooya and Pakdaman (2019)	"	Continuous optimal control				*				
Singer and Khmelnitsky (2021)		Continuous optimal control		*		*		*		*
Dhaiban (2022)		Continuous optimal control		*		*	*	*	*	*
Dong et al. (2011)	"	Continuous optimal control , multi-stage								*
Hatami-Marbini et al. (2020)		Continuous optimal control				*		*	*	*
Benkherouf et al. (2016)	"	Continuous optimal control		*				*		
Vercraene and Gayon (2013)	"	Discrete optimal control, two-stage	*					*		*
Lee et al. (2021)		Continuous optimal control		*		*		*		*
Ignaciuk and Bartoszewicz (2010)	"	Discrete optimal control , multi-stage	*							*
Al-Khazraji et al. (2021)	"	Continuous optimal control								*

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			Model Challenge							
Author	**Topic**	**Type of Model**	**Discrete Time**	**No possibility of reworking**	**One-stage**	**Without LT**	**Infinite capacity**	**No possibility of recycling**	**No optimization**	**BOM is not allowed**
Öztürk (2021)	"	Continuous optimal control		*		*			*	*
ElHafsi et al. (2021)	"	Continuous optimal control				*			*	*
Gao (2022)	"	Continuous optimal control				*			*	*
Shen et al. (2022)	"	Continuous optimal control				*			*	*
Rachih et al. (2022)	"	Discrete optimal control	*	*		*				*
Megoze et al. (2022)	"	Continuous optimal control		*		*			*	*
Ignaciuk (2022)	"	Discrete optimal control	*			*			*	*
Yan and Sun (2024)	MRP: Optimal control applications in production planning and inventory control	Continuous optimal control, multi-stage, stochastic	*			*	*			*
Luo et al., 2024	Optimal control applications in production planning and inventory control	Continuous optimal control, multi-stage	*	*			*	*		*
present study	"	Continuous optimal control, multi-stage	Continuous	possibility of reworking	multi-stage	Existence of LT	Finite capacity	possibility of recycling	Optimization	Existence of BOM

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2. Model descriptions

The proposed model in this study will be a multi-stage production-inventory planning model that involves returning, recycling, and reworking the returned products. Since the proposed model will consider the production lead time in each stage, two models are presented. In the first model, the production lead time will be zero, and in the second model, the actual lead time will be considered at each stage of production. The intended production-inventory system produces a final product called A. In the first stage of the model, two raw materials (b and c) will be bought and assembled. Then, the work in process (A_1) will be produced. In the second stage, the work in process will be mixed with the raw material (d), and the final item (A) will be produced. To produce each unit of (A_1), we need α unit of b and β unit of c. In the next stage, to make each unit of A, we need δ unit of A_1 and γ unit of d (figure 1).

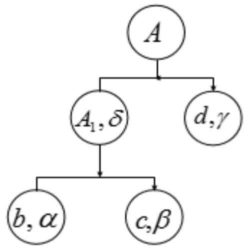

Figure 1. product three-level tree

Pooya and Pakdaman (2019) presented an MRP optimal control model for finite capacity without considering the lead time. In another article (2018), they offered a multi-stage model considering production lead time. The proposed model in this study combines these two models. The multi-stage production-inventory system shown in Figure 2 is a continuous MRP model, including the stages of recycling and reworking on returned items. In this model, the disposal stage of deteriorating items is also considered. In the proposed model above, the first step is ordering the raw materials (b and c), which is shown with $s_{mb}(t)$ and $s_{mc}(t)$, respectively. The amount of these two raw materials in the unit of time also equals to $I_{mb}(t)$ and $I_{mc}(t)$. Next, by combining the two raw materials, the work in process (A_1) is produced. The amount of shipment of the above items for production equals to $P_{mb}(t)$ and $P_{mc}(t)$, respectively. The first stage inventory ($I_{mA_1}(t)$) will be divided into two parts. Part of the inventory will be sent to the next stage ($P_{mA_1}(t)$) for A production and the other part will be returned at a ω_{m1} rate. Some of the returned goods will be shipped to the remanufacturing or reworking stage at a ω_{RA_1}rate and some will be referred to the recycling stage at a ω_{CA_1} rate (from the inventory of recycled goods, the part related to the raw material b ($C_{rb}(t)$)will be added to the inventory of b and part related to the raw material (c) will be added to the inventory of c).

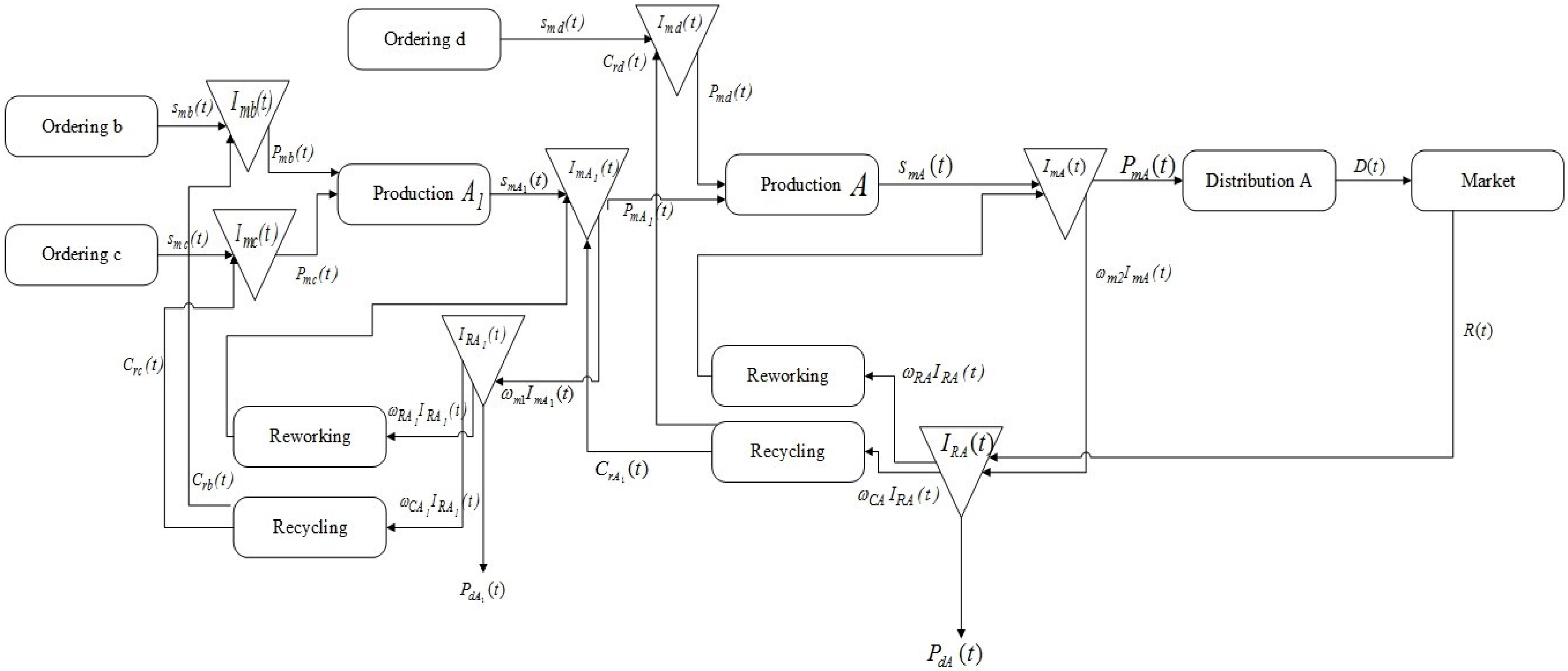

Figure 2. Diagram of the proposed model

The remaining items at this stage that cannot be reworked or recycled will be disposed of at a ω_{dA_1}rate. In the second stage of production, the inventory of manufactured goods ($I_{mA}(t)$) will be divided into two parts. Part of the inventory will be shipped to the market based on the market demand ($D\ (t)$) (that $P_{mA}\ (t) = D(t)$) and the remainder will constitute returned goods at the rate (ω_{RA}) and returned goods, similar to the preceding two-stage, will be added to the remanufacturing (ω_{dA}) or recycling stage (at the rate ω_{CA}) and some of them will be disposed of at the ω_{dA} rate. Some of the shipped goods will also be returned to the market ($R\ (t)$) which will be added to the second stage returned goods inventory.

The variables and parameters used in the model will be as follows. These symbols are used throughout the article for state and control variables.

State and control variables

$I_{mb}(t)$: Inventory of ordered items b at time t
$I_{mc}(t)$: Inventory of ordered items c at time t
$I_{md}(t)$: Inventory of ordered items d at time t
$I_{mA_1}(t)$: Inventory of ordered items A_1 at time t
$I_{mA}(t)$: Inventory of manufactured final production A at time t
$I_{RA_1}(t)$: Inventory of returned items A_1 at time t
$I_{RA}(t)$: Inventory of returned final production A at time t

Control variables

$s_{mb}(t)$: Level of ordering and releasing the items b at time t
$s_{mc}(t)$: Level of ordering and releasing the items C at time t
$s_{md}(t)$: Level of ordering and releasing the items d at time t
$s_{mA_1}(t)$: Level of ordering and releasing the WIP items A_1at time t
$s_{mA}(t)$: Level of ordering and releasing the final production A at time t
$P_{mb}(t)$: Level of items b which will be transmitted for producing A_1at time t
$P_{mc}(t)$: Level of items c which will be transmitted for producing A_1at time t
$P_{md}(t)$: Level of items d which will be transmitted for producing A at time t
$P_{mA_1}(t)$: Level of WIP items A_1 which will be transmitted for producing A at time t
$P_{mA}(t)$: Level of manufactured and released items A at time t
$C_{rb}(t)$: Level of recycled items b at time t
$C_{rc}(t)$: Level of recycled items c at time t
$C_{rd}(t)$: Level of recycled items d at time t
$C_{rA_1}(t)$: Level of recycled WIP items A_1 at time t
$D\ (t)$: Released demand of manufactured items A at time t
$R\ (t)$: Level of returned items A from market at time t

Parameters

ω_{m1}: Rate of returning in the stock of manufactured WIP items A_1
ω_{m2}:Rate of returning in the stock of manufactured final production A
ω_{RA_1}: Rate of reworking in the stock of returned WIP items A_1

ω_{RA}: Rate of reworking in the stock of returned final production A
ω_{CA_1}: Rate of recycling in the stock of returned WIP items A_1
ω_{CA}: Rate of recycling in the stock of returned final production A
ω_{dA_1}: Deterioration rate in the stock of reworked WIP items A_1
ω_{dA}:Deterioration rate in the stock of reworked final production A
α: Number of items b which will be used for producing A_1
β: Number of items c which will be used for producing A_1
γ :Number of items d which will be used for producing A
δ: Number of item A_1 which will be used for producing A
τ_{A_1}: Production Lead Time for WIP items A_1
τ_A: Production Lead Time for final production A

The number of shipped items (b and c) should be a coefficient of their use in the BOM as otherwise, the amount of raw material shipped for production A_1 would be excessive. This coefficient is defined a. In other words, the number of goods shipped at any time should be such that they are all used for production A_1 and there is no additional shipment of any of them. It should also be a coefficient on the amount of inventory available at any given time. So, it can be stated:

$$\begin{cases} P_{mb}(t) = a\alpha I_{mb}(t) \\ P_{mc}(t) = a\beta I_{mc}(t) \end{cases} \tag{1}$$

But given that the number of items shipped at any time should be less than the number of goods above, so it should be $0 < a\alpha, a\beta \leq 1$.

The order of manufactured item (A_1) at any given time is obtained from the following relationship:

$$s_{mA_1}(t) = min\left\{\frac{P_{mb}(t)}{\alpha}, \frac{P_{mc}(t)}{\beta}\right\} \tag{2}$$

In order not to ship items b and c in production A_1 more than the production need, it should be:

$$\frac{P_{mb}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} \tag{3}$$

So it can be stated:

$$s_{mA_1}(t) \min\left\{\frac{P_{mb}(t)}{\alpha}, \frac{P_{mc}(t)}{\beta}\right\} = \frac{P_{mb}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} \tag{4}$$

According to the relationships (1):

$$\begin{cases} P_{mb}(t) = a\alpha I_{mb}(t) \Longrightarrow \frac{P_{mb}(t)}{\alpha} = aI_{mb}(t) \\ P_{mc}(t) = a\beta I_{mc}(t) \Longrightarrow \frac{P_{mc}(t)}{\beta} = aI_{mc}(t) \end{cases} \tag{5}$$

According to the relationship (3):

$$aI_{mb}(t) = aI_{mc}(t) \Longrightarrow I_{mb}(t) = I_{mc}(t) \quad (6)$$

In the next stage, the manufactured product will be transformed into the final product (A) by mixing the raw material (d). At this point, the shipping rate of the above items is equal to $P_{md}(t)$ and $P_{mA_1}(t)$, respectively. Similar to the one described in the previous stage, we can conclude the following relationships (here the coefficient $\acute{b}$ is defined):

$$\begin{aligned} P_{md}(t) &= \acute{b}\gamma I_{md}(t) \\ P_{mA_1}(t) &= \acute{b}\delta I_{mA_1}(t) \end{aligned} \quad (7)$$

In which $0 < \acute{b}\gamma, \acute{b}\delta \leq 1$.

At this point, the manufactured product order A must follow the following relationship at all times.

$$s_{mA}(t) = \min \frac{P_{md}(t)}{\gamma}, \frac{P_{mA_1}(t)}{\delta} \quad (8)$$

In order not to ship items d and A_1 in production A more than the production need, it should be:

$$\frac{P_{md}(t)}{\gamma} = \frac{P_{mA_1}(t)}{\delta} \quad (9)$$

And according to the relationships (9) and similar to the above, it can be stated: $I_{mA_1}(t) = I_{md}(t)$

By placing relationships (3) and (9) in relationships (2) and (8), the following relationships will be achieved:

$$\begin{cases} s_{mA_1}(t) = \min\left\{\frac{P_{md}(t)}{\alpha}, \frac{P_{mc}(t)}{\beta}\right\} = \frac{P_{md}(t)}{\alpha} = \frac{P_{mc}(t)}{\beta} = \frac{1}{2\alpha}P_{mb}(t) + \frac{1}{2\beta}P_{mc}(t) \\ s_{mA}(t) = \min\left\{\frac{P_{md}(t)}{\gamma}, \frac{P_{mA_1}(t)}{\delta}\right\} = \frac{P_{md}(t)}{\gamma} = \frac{P_{mA_1}(t)}{\delta} = \frac{1}{2\gamma}P_{mb}(t) + \frac{1}{2\gamma}P_{mA_1}(t) \end{cases} \quad (10)$$

And according to Figure 2,and recycling and detriration rates it can be stated:

$$\begin{cases} C_{rb}(t) = \omega_{CA_1} I_{RA_1}(t) \\ C_{rc}(t) = \beta\omega_{CA_1} I_{RA_1}(t) \\ C_{rd}(t) = \gamma\omega_{CA_2} I_{RA}(t) \\ C_{rA_1}(t) = \delta\omega_{CA_2} I_{RA}(t) \end{cases}$$

$$\begin{cases} P_{dA_1}(t) = \omega_{dA_1} I_{RA_1}(t) \\ P_{dA}(t) = \omega_{dA} I_{RA}(t) \end{cases} \quad (11)$$

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2.1. Model one: The mode without considering the lead time

The dynamic behavior of the above-proposed model can be expressed in terms of the following relationships:

$$\begin{cases} I_{mb}(t) = s_{mb}(t) + C_{rb}(t) - P_{mb}(t) & I_{mb}(0) = I_{mb}^{0} \\ I_{mc}(t) = s_{mc}(t) + C_{rc}(t) - P_{mc}(t) & I_{mc}(0) = I_{mc}^{0} \\ I_{mA_1}(t) = s_{mA_1}(t) + \omega_{RA_1} I_{RA_1}(t) + C_{rA_1}(t) - P_{mA_1}(t) - \omega_{m_1} I_{mA_1}(t) & I_{mA_1}(0) = I_{mA_1}^{0} \\ I_{md}(t) = s_{md}(t) + C_{rd}(t) - P_{md}(t) & I_{md}(0) = I_{md}^{0} \\ I_{mA}(t) = s_{mA}(t) + \omega_{RA} I_{RA}(t) - P_{mA}(t) - \omega_{m_2} I_{mA}(t) & I_{mA}(0) = I_{mA}^{0} \\ I_{RA_1}(t) = \omega_{m_1} I_{mA_1}(t) - \omega_{RA_1} I_{RA_1}(t) - \omega_{CA_1} I_{RA_1}(t) - P_{dA_1}(t) & I_{RA_1}(0) = I_{RA_1}^{0} \\ I_{RA}(t) = \omega_{m_2} I_{mA}(t) + R(t) - \omega_{RA_2} I_{RA_1}(t) - \omega_{CA_2} I_{RA}(t) - P_{dA}(t) & I_{RA}(0) = I_{RA}^{0} \end{cases} \tag{12}$$

By placing the relationships category (10) and (11) into the relationship category (12), the following relationships are obtained:

$$\begin{cases} I_{mb}(t) = s_{mb}(t) + \alpha\omega_{CA_1} I_{RA_1}(t) - P_{mb}(t) & I_{mb}(0) = I_{mb}^{0} \\ I_{mc}(t) = s_{mc}(t) + \beta\omega_{CA_1} I_{RA_1}(t) - P_{mc}(t) & I_{mc}(0) = I_{mc}^{0} \\ I_{mA_1}(t) = \dfrac{1}{2\alpha} P_{mb}(t) + \dfrac{1}{2\beta} P_{mc}(t) + \omega_{RA_1} I_{RA_1}(t) + \gamma\omega_{CA_2} I_{RA_2}(t) - P_{mA_1}(t) - \omega_{m_1} I_{mA_1}(t) & I_{mA_1}(0) = I_{mA_1}^{0} \\ I_{md}(t) = s_{md}(t) + \gamma\omega_{CA} I_{RA}(t) - P_{md}(t) & I_{md}(0) = I_{md}^{0} \\ I_{mA}(t) = \dfrac{1}{2\gamma} P_{md}(t) + \dfrac{1}{2\delta} P_{mA_1}(t) + \omega_{RA} I_{RA}(t) - P_{mA}(t) - \omega_{m_2} I_{mA}(t) & I_{mA}(0) = I_{mA}^{0} \\ I_{RA_1}(t) = \omega_{m_1} I_{mA_1}(t) - \omega_{RA_1} I_{RA_1}(t) - \omega_{CA_1} I_{RA_1}(t) - \omega_{dA_1} I_{RA_1}(t) & I_{RA_1}(0) = I_{RA_1}^{0} \\ I_{RA}(t) = \omega_{m_2} I_{mA}(t) + R(t) - \omega_{RA} I_{RA_1}(t) - \omega_{CA} I_{RA}(t) - \omega_{dA} I_{RA}(t) & I_{RA}(0) = I_{RA}^{0} \end{cases} \tag{13}$$

Column matrices ($\tilde{x}(t)$ and $\tilde{u}(t)$) including state and control variables are defined as follows:

$$\tilde{x}(t) = [I_{mb}(t)\ I_{mc}(t)\ I_{mA_1}(t)\ I_{md}(t)\ I_{mA}(t)\ I_{RA_1}(t)\ I_{RA}(t)]^{T}$$

$$\tilde{u}(t) = [s_{mb}(t)\ s_{mc}(t)\ s_{md}(t)\ P_{mb}(t)\ P_{mc}(t)\ P_{mA_1}(t)\ P_{md}(t)\ P_{mA}(t)\ R(t)]^{T}$$

The matrix A whose entries form the coefficients of state variables in the relationships category (13), is defined as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha\omega_{CA_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta\omega_{CA_1} & 0 \\ 0 & 0 & -\omega_{m1} & 0 & 0 & \omega_{RA_1} & \delta\omega_{CA} \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma\omega_{CA} \\ 0 & 0 & 0 & 0 & -\omega_{m2} & 0 & \omega_{RA} \\ 0 & 0 & \omega_{m1} & 0 & 0 & -(\omega_{RA_1} + \omega_{CA_1} + \omega_{dA_1}) & 0 \\ 0 & 0 & 0 & 0 & \omega_{m2} & 0 & -(\omega_{RA} + \omega_{CA} + \omega_{dA}) \end{bmatrix}$$

And the matrix B whose entries form the coefficients of control variables in the relationships category (13), is defined as follows:

$$B=\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\alpha} & \frac{1}{2\beta} & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\delta} & \frac{1}{2\gamma} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

According to A and B matrices, it could be stated:

$$\dot{\tilde{x}}\,(t) = A\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{x}(0)\tilde{x}_0 \tag{14}$$

To obtain the target values of the state and control variables, the relationship is defined as follows:

$$\Delta f\,(t) = f(t) - \hat{f}\,(t) \tag{15}$$

Where f is the control or state variable and $\hat{f}$ is the target value of f. So we define $x(t)$ and $u(t)$ matrices as:

$$x(t) = [\Delta I_{mb}(t)\ \ \Delta I_{mc}(t)\ \ \Delta I_{mA_1}(t)\ \Delta I_{md}(t)\ \ \Delta I_{mA}(t)\ \ \Delta I_{RA_1}(t)\ \Delta I_{RA}(t)]^T$$

$$u(t) = [\Delta s_{mb}(t)\ \ \Delta s_{mc}(t)\ \ \Delta s_{md}(t)\ \ \Delta P_{mb}(t)\ \ \Delta P_{mc}(t)\ \ \Delta P_{mA_1}\ \Delta P_{md}(t)\ \ \Delta P_{mA}(t)\ \ \Delta R(t)]^T$$

The goals for the state and control variables are the constraints that apply to these variables in the model. The purpose of producing, remanufacturing, or recycling items is the finite capacity for which the workstation is intended, and the purpose of inventories is the finite capacity that should be considered for warehouses. Therefore, when the objective function is a minimization type of Δf_i, the objective function is actually to converge between the control and the state variables with their target values.

Now, considering the above explanation and the set of equations (13), the problem of linear binomial optimal control with finite time is defined as follows:

$$min\ k_1(\Delta I_{mb}(T))^2 + k_2(\Delta I_{mc}(T))^2 + k_3(\Delta I_{mA_1}(T))^2 + k_4(\Delta I_{md}(T))^2 + k_5(\Delta I_{mA}(T))^2 + k_6(\Delta I_{RA_1}(T))^2 + k_7(\Delta I_{RA}(T))^2$$

$$\int_0^T [q_1\left(\Delta I_{mb}(t)\right)^2 + q_{2(\Delta I_{mc}(t))^2} + q_3\left(\Delta I_{mA_1}(t)\right)^2 + q_4\left(\Delta I_{md}(t)\right)^2 + q_5\left(\Delta I_{mA}(t)\right)^2 + q_6\left(\Delta I_{RA_1}(t)\right)^2 + q_7\left(\Delta I_{RA}(t)\right)^2 + r_1\Delta s_{mb}(t) + r_2\Delta s_{mc}(t) + r_3\Delta s_{md}(t) + r_4\Delta P_{mb}(t) + r_5\Delta P_{mc}(t) + r_6\Delta P_{mA_1}(t) + r_7\Delta P_{md}(t) + r_8\Delta P_{mA}(t) + r_9\Delta R(t)]dt$$

$St.$ (16)

$$\begin{cases} \Delta\dot{I}_{mb}(t) = \Delta s_{mb}(t) + \alpha\omega_{CA_1}\Delta I_{RA_1}(t) - \Delta P_{mb}(t) & \Delta I_{mb}(0) = \Delta I^0_{mb} \\ \Delta\dot{I}_{mc}(t) = \Delta s_{mc}(t) + \beta\omega_{CA_1}\Delta I_{RA_1}(t) - \Delta P_{mc}(t) & \Delta I_{mc}(0) = \Delta I^0_{mc} \\ \Delta\dot{I}_{mA_1}(t) = \frac{1}{2\alpha}\Delta P_{mb}(t) + \frac{1}{2\beta}\Delta P_{mc}(t) + \omega_{RA_1}\Delta I_{RA_1}(t) + \delta\omega_{CA_2}\Delta I_{RA_2}(t) - \Delta P_{mA_1}(t) - \omega_{m_1}\Delta I_{mA_1}(t) & \Delta I_{mA_1}(0) = \Delta I^0_{mA_1} \\ \Delta\dot{I}_{md}(t) = \Delta s_{md}(t) + \gamma\omega_{CA}\Delta I_{RA}(t) - \Delta P_{md}(t) & \Delta I_{md}(0) = \Delta I^0_{md} \\ \Delta\dot{I}_{mA}(t) = \frac{1}{2\gamma}\Delta P_{md}(t) + \frac{1}{2\delta}\Delta P_{mA_1}(t) + \omega_{RA}\Delta I_{RA}(t) - \Delta P_{mA}(t) - \omega_{m_2}\Delta I_{mA}(t) & \Delta I_{mA}(0) = \Delta I^0_{mA} \\ \Delta\dot{I}_{RA_1}(t) = \omega_{m_1}\Delta I_{mA_1}(t) - \omega_{RA_1}\Delta I_{RA_1}(t) - \omega_{CA_1}\Delta I_{RA_1}(t) - \omega_{dA_1}\Delta I_{RA_1}(t) & \Delta I_{RA_1}(0) = \Delta I^0_{RA_1} \\ \Delta\dot{I}_{RA}(t) = \omega_{m_2}\Delta I_{mA}(t) + \Delta R(t) - \omega_{RA}\Delta I_{RA_1}(t) - \omega_{CA}\Delta I_{RA}(t) - \omega_{dA}\Delta I_{RA}(t) & \Delta I_{RA}(0) = \Delta I^0_{RA} \end{cases}$$

Where k_i and q_i $(i = (1,2,\ldots,7))$ and also r_i $(j = (1,2,\ldots,11))$ are non-negative real numbers that are considered as penalty coefficients of the deviation of the variables from their target values.

As is known, the matrix entries A and B are coefficients of the state and control variables of the model constraints (16).

2.2. Model two: The mode with considering the lead time

In this model, the lead time is considered in each stage of production. Therefore, the production lead time for the work in process (A_1) is τ_{A_1} and for Product A is defined as τ_A.

$$I_{mA_1}(t) = a\alpha^2 I_{mb}(t-\tau_{A_1}) + a\beta^2 I_{mc}(t-\tau_{A_1}) + \omega_{RA_1}I_{RA_1}(t) + \delta\omega_{CA}I_{RA}(t) - P_{mA_1}(t) - \omega_{m_1}I_{mA_1}(t) \quad I_{mA_1}(0) = I^0_{mA_1} \tag{17}$$

The following relationship will be obtained considering the lead time for the product A:

$$\dot{I}_{mA}(t) = \frac{\acute{b}}{2}I_{md}(t-\tau_A) + \frac{\acute{b}}{2}I_{mA_1}(t-\tau_A) + \omega_{RA}I_{RA}(t) - P_{mA}(t) - \omega_{m_2}I_{mA}(t) \quad I_{mA}(0) = I^0_{mA} \tag{18}$$

Since the above two equations are a linear system of delayed differential equations, we will have the equation (13) using Taylor expansion as follows:

$$\begin{cases} \dot{I}_{mb}(t) = s_{mb}(t) + \alpha\omega_{CA_1}I_{RA_1}(t) - P_{mb}(t) \quad I_{mb}(0) = I^0_{mb} \\ \dot{I}_{mc}(t) = s_{mc}(t) + \beta\omega_{CA_1}I_{RA_1}(t) - P_{mc}(t) \quad I_{mc}(0) = I^0_{mc} \\ \dot{I}_{mA_1}(t) = a\alpha^2(I_{mb}(t) - \tau_{A_1}\dot{I}_{mb}(t)) + a\beta^2\left(I_{mc}(t) - \tau_{A_1}\dot{I}_{mc}(t)\right) + \omega_{RA_1}I_{RA_1}(t) + \delta\omega_{CA}I_{RA}(t) \\ -P_{mA_1}(t) - \omega_{m_1}I_{mA_1}(t) \quad I_{mA_1}(0) = I^0_{mA_1} \\ \dot{I}_{md}(t) = s_{md}(t) + \gamma\omega_{CA}\Delta I_{RA}(t) - b\gamma I_{md}(t) \quad I_{md}(0) = I^0_{md} \\ \dot{I}_{mA}(t) = \frac{\acute{b}}{2}\left(I_{md}(t) - \tau_A\dot{I}_{md}(t)\right) + \frac{\acute{b}}{2}\left(I_{mA_1}(t) - \tau_A\dot{I}_{mA_1}(t)\right) + \omega_{RA}I_{RA}(t) - P_{mA}(t) - \omega_{m_2}I_{mA}(t) \quad I_{mA}(0) = I^0_{mA} \\ \dot{I}_{RA_1}(t) = \omega_{m_1}\Delta I_{mA_1}(t) - \omega_{RA_1}\Delta I_{RA_1}(t) - \omega_{CA_1}I_{RA_1}(t) - \omega_{dA_1}\Delta I_{RA_1}(t) \quad I_{RA_1}(0) = I^0_{RA_1} \\ \dot{I}_{RA}(t) = \omega_{m_2}I_{mA}(t) + R(t) - \omega_{RA}I_{RA_1}(t) - \omega_{CA}I_{RA}(t) - \omega_{dA}I_{RA}(t) \quad I_{RA}(0) = I^0_{RA} \end{cases} \tag{19}$$

The above equation will be converted as follows after simplification, by placing the first an d second equations in the third equation:

$$\begin{aligned}\dot{I}_{mA_1}(t) = {} & (a\alpha^2 + \tau_{A_1} a^2\alpha^3) I_{mb}(t) - a\alpha^2\tau_{A_1}(s_{mb}(t) + (-\tau_{A_1} a\alpha^3\omega_{CA_1} - a\beta^3\tau_{A_1}\omega_{CA_1} \\ & + \omega_{RA_1}) I_{RA_1}(t) + (a\beta^2 \\ & + \tau_{A_1} a^2\beta^3) I_{mc}(t) - \tau_{A_1} a\beta^2 (s_{mc}(t) + \delta\omega_{CA} I_{RA}(t) - P_{mA_1}(t) \\ & - \omega_{m1} I_{mA_1}(t) \quad I_{mA_1}(0) = I^0_{mA_1}\end{aligned} \tag{20}$$

And putting the equation (20) and the fourth equation of the equation group (19) in the fifth equation will transform the above equation as follows after simplification:

$$\begin{aligned}\dot{I}_{mA}(t) = {} & \left(\frac{\acute{b}}{2} - \frac{\acute{b}^2}{2}\gamma\tau_A\right) I_{md}(t) - \frac{\acute{b}}{2}\tau_A s_{md}(t) + \left(\frac{\acute{b}}{2}\gamma\omega_{CA}\tau_A - \frac{\acute{b}}{2}\tau_A\delta\omega_{CA} + \omega_{RA}\right) I_{RA}(t) \\ & + \left(\frac{\acute{b}}{2} + \frac{\acute{b}^2}{2}\tau_A\omega_{m_1}\right) I_{mA_1}(t) - \frac{\acute{b}}{2}\tau_A\left(a\alpha^2 + \tau_{A_1} a^2\alpha^3\right) I_{mb}(t) \\ & + \frac{\acute{b}}{2}\tau_A a\alpha^2\tau_{A_1} s_{mb}(t) + \frac{\acute{b}}{2}\tau_A\left(\tau_{A_1} a\alpha^3\omega_{CA_1} + a\beta^3\tau_{A_1}\omega_{CA_1} - \omega_{RA_1}\right) I_{RA_1}(t) \\ & - \frac{\acute{b}}{2}\tau_A(\, a\beta^2 + \tau_{A_1}\, a^2\beta^3\,) I_{mC}(t) + \frac{\acute{b}}{2} a\beta^2\tau_A\tau_{A_1} s_{mC}(t) + \frac{\acute{b}}{2}\tau_A\, P_{mA_1}(t) \\ & - P_{mA}(t) - \omega_{m_2}\, I_{mA}(t) \qquad I_{mA}(0) = I^0_{mA}\end{aligned} \tag{21}$$

Therefore, by placing equations (20) and (21), the equation groups (19) will be as follows:

$$\begin{cases} \dot{I}_{mb}(t) = s_{mb}(t) + \alpha\omega_{CA_1} I_{RA_1}(t) - P_{mb}(t) \qquad I_{mb}(0) = I^0_{mb} \\ \dot{I}_{mc}(t) = s_{mc}(t) + \beta\omega_{CA_1} I_{RA_1}(t) - P_{mc}(t) \qquad I_{mc}(0) = I^0_{mc} \\ \dot{I}_{mA_1}(t) = \ (a\alpha^2 + \tau_{A_1} a^2\alpha^3) I_{mb}(t) - a\alpha^2\tau_{A_1} s_{mb}(t) + (-\tau_{A_1} a\alpha^3\omega_{CA_1} - a\beta^3\tau_{A_1}\omega_{CA_1} + \omega_{RA_1}) I_{RA_1}(t) \\ +(a\beta^2 + \tau_{A_1} a^2\beta^3) I_{mc}(t) - a\beta^2\tau_{A_1} s_{mc}(t) + \delta\omega_{CA} I_{RA}(t) - P_{mA_1}(t) - \omega_{m1} I_{mA_1}(t) \\ \dot{I}_{md}(t) = s_{md}(t) + \gamma\omega_{CA} I_{RA}(t) - P_{md}(t) \qquad I_{md}(0) = I^0_{md} \\ \dot{I}_{mA}(t) = \left(\frac{\acute{b}}{2} - \frac{\acute{b}^2}{2}\gamma\tau_A\right) I_{md}(t) - \frac{\acute{b}}{2}\tau_A s_{md}(t) + \left(\frac{\acute{b}}{2}\gamma\omega_{CA}\tau_A - \frac{\acute{b}}{2}\tau_A\delta\omega_{CA} + \omega_{RA}\right) I_{RA}(t) \\ + \left(\frac{\acute{b}}{2} + \frac{\acute{b}^2}{2}\tau_A\omega_{m_1}\right) I_{mA_1}(t) - \frac{\acute{b}}{2}\tau_A\left(a\alpha^2 + \tau_{A_1} a^2\alpha^3\right) I_{mb}(t) \\ + \frac{\acute{b}}{2}\tau_A a\alpha^2\tau_{A_1} s_{mb}(t) + \frac{\acute{b}}{2}\tau_A\left(\tau_{A_1} a\alpha^3\omega_{CA_1} + a\beta^3\tau_{A_1}\omega_{CA_1} - \omega_{RA_1}\right) I_{RA_1}(t) \\ - \frac{\acute{b}}{2}\tau_A(\, a\beta^2 + \tau_{A_1}\, a^2\beta^3\,) I_{mC}(t) + \frac{\acute{b}}{2} a\beta^2\tau_A\tau_{A_1} s_{mC}(t) + \frac{\acute{b}}{2}\tau_A\, P_{mA_1}(t) - P_{mA}(t) - \omega_{m_2}\, I_{mA}(t) \quad I_{mA}(0) = I^0_{mA} \\ \dot{I}_{RA_1}(t) = \omega_{m_1} I_{mA_1}(t) - \omega_{RA_1} I_{RA_1}(t) - \omega_{CA_1} I_{RA_1}(t) - \omega_{dA_1} I_{RA_1}(t) \qquad I_{RA_1}(0) = I^0_{RA_1} \\ \dot{I}_{RA}(t) = \omega_{m_2} I_{mA}(t) + R(t) - \omega_{RA} I_{RA_1}(t) - \omega_{CA} I_{RA}(t) - \omega_{dA} I_{RA}(t) \qquad I_{RA}(0) = I^0_{RA} \end{cases} \tag{22}$$

The column matrices ($\tilde{x}(t)$and $\tilde{u}(t)$), which include the state and control variables, are defined as follows:

$\tilde{x}(t) = [I_{mb}(t)\ \ I_{mc}(t)\ \ I_{mA_1}(t)\ I_{md}(t)\ \ I_{mA}(t)\ \ I_{RA_1}(t)\ I_{RA}(t)]^T$

$\tilde{u}(t) = [s_{mb}(t)\ \ s_{mc}(t)\ \ s_{md}(t)\ \ P_{mb}(t)\ \ P_{mc}(t)\ \ P_{mA_1}\ P_{md}(t)\ \ P_{mA}(t)\ \ R(t)]^T$

The matrix $\acute{A}$ whose entries form the coefficients of state variables in the relationships category (27), is defined as follows:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha\omega_{CA_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta\omega_{CA_1} & 0 \\ (a\alpha^2+\tau_{A_1}a^2\alpha^3) & (a\beta^2+\tau_{A_1}a^2\beta^3) & -\omega_{m1} & 0 & 0 & -(\tau_{A_1}a\alpha^3\omega_{CA_1}+a\beta^3\tau_{A_1}\omega_{CA_1}-\omega_{RA_1}) & \delta\omega_{CA} \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma\omega_{CA} \\ -\frac{b'}{2}\tau_A(a\alpha^2+\tau_{A_1}a^2\alpha^3) & -\frac{b'}{2}\tau_A(a\beta^2+\tau_{A_1}a^2\beta^3) & (\frac{b'}{2}+\frac{b'}{2}\tau_A\omega_{m1}) & (\frac{b'}{2}-\frac{b'^2}{2}\gamma\tau_A) & -\omega_{m2} & \frac{b'}{2}\tau_A(\tau_{A_1}a\alpha^3\omega_{CA_1}+a\beta^3\tau_{A_1}\omega_{CA_1}-\omega_{RA_1}) & (\frac{b'}{2}\gamma\omega_{CA}\tau_A-\frac{b'}{2}\tau_A\delta\omega_{CA}+\omega_{RA}) \\ 0 & 0 & \omega_{m1} & 0 & 0 & -(\omega_{RA_1}+\omega_{CA_1}+\omega_{dA_1}) & 0 \\ 0 & 0 & 0 & 0 & \omega_{m2} & 0 & -(\omega_{RA}+\omega_{CA}+\omega_{dA}) \end{bmatrix}$$

The matrix $\acute{B}$ whose entries form the coefficients of control variables in the relationships category (27), is defined as follows:

$$B' = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -a\alpha^2\tau_{A_1} & -a\beta^2\tau_{A_1} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \frac{b'}{2}a\alpha^2\tau_A\tau_{A_1} & \frac{b'}{2}a\beta^2\tau_A\tau_{A_1} & -\frac{b'}{2}\tau_A & 0 & 0 & \frac{b'}{2}\tau_A & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Model solving

In this section, the system of differential equations related to the proposed model will be derived and then solved.

If $Q = diag\{q_i\}, K = diag\{k_i\}$ and $R = diag\{r_j\}$ is defined for $i = 1,2, \dots ,7$ and also $j = 1,2, \dots ,9$ is defined as diagonal matrices that are penalty factors, the model (15) can then be summarized as follows:

$$\begin{aligned} &Minimizexx\ (T)^t Kx\ (T) + \int_0^T [x\ (t)^t Qx\ (t) + u(t)^t Ru\ (t) d(t) \\ &st \\ &\dot{x}\ (t) = Ax(t) + Bu\ (t), \qquad x(0) = (x)^0 \end{aligned} \tag{23}$$

Where K and Q are both the real diagonal matrices from the 7th grade and R is the real diagonal matrices from the 9th grade with positive entries.

Adequate optimization conditions are usually obtained for optimal control problems using the Hamiltonian. Hamiltonian should first be identified as follows to achieve optimum, sufficient conditions:

$$H(x(t), u(t), \lambda(t), t) = x(t)^t Qx\ (t) + u(t)^t Ru\ (t) + \lambda(t)^t [Ax\ (t) + Bu\ (t)] \tag{24}$$

Based on the maximum principle, the optimum conditions for the problem (23) are as follows:

$$\begin{aligned}
&\frac{\partial H(x(t),u(t),\lambda(t),t)}{\partial x} = \dot{\lambda}(t), \lambda(T) = 2Kx(T)\\
&\frac{\partial H(x(t),u(t),\lambda(t),t)}{\partial \lambda} = \dot{x}(t), x(0) = x^0\\
&\frac{\partial H(x(t),u(t),\lambda(t),t)}{\partial u} = 0
\end{aligned} \tag{25}$$

Or its equivalent can be written as follows:

$$\begin{aligned}
&-\dot{\lambda}(t) = 2Qx(t) + A^t\lambda(t), \qquad \lambda(T) = 2Kx(T)\\
&\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x^0\\
&u(t) = -\frac{1}{2}R^{-1}B^t\lambda(t)
\end{aligned} \tag{26}$$

From the third equation of the equation category (26), the optimal control law can be obtained. Therefore, the system of differential equations (26) is converted as follows:

$$\begin{aligned}
&\dot{\lambda}(t) = -(2Qx(t) + A^t\lambda(t)), \qquad \lambda(T) = 2Kx(T)\\
&\dot{x}(t) = Ax(t) - \frac{1}{2}BR^{-1}B^t\lambda(t) \;\; x(0) = x^0
\end{aligned} \tag{27}$$

It should be noted that the equations category (26) as well as (27) are optimum conditions. To ensure that the control obtained $u^*(t)$ meets the required optimization conditions, which is optimize the minimization of the problem (23), the following matrix must be defined positively (Subbaram Naido, 2002).

$$\Pi = \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \tag{28}$$

Which may be written as follows:

$$\Pi = \begin{bmatrix} 2Q & 0 \\ 0 & 2R \end{bmatrix}$$

However, since diagonal matrices (R and Q) have positive entries, the matrix Π is positive and therefore the control obtained $u^*(t)$ has the necessary optimization conditions.

4. Numerical simulation

The model was used by the Mashhad Panel Barsava Manufacturing Company to validate the proposed models and assess their applicability. The company's 3D panels are simulated in Figure 3, in the three-stage production-inventory system. MATLAB 2017 software will solve the final optimal control model to obtain the necessary results and objectives set for the research.

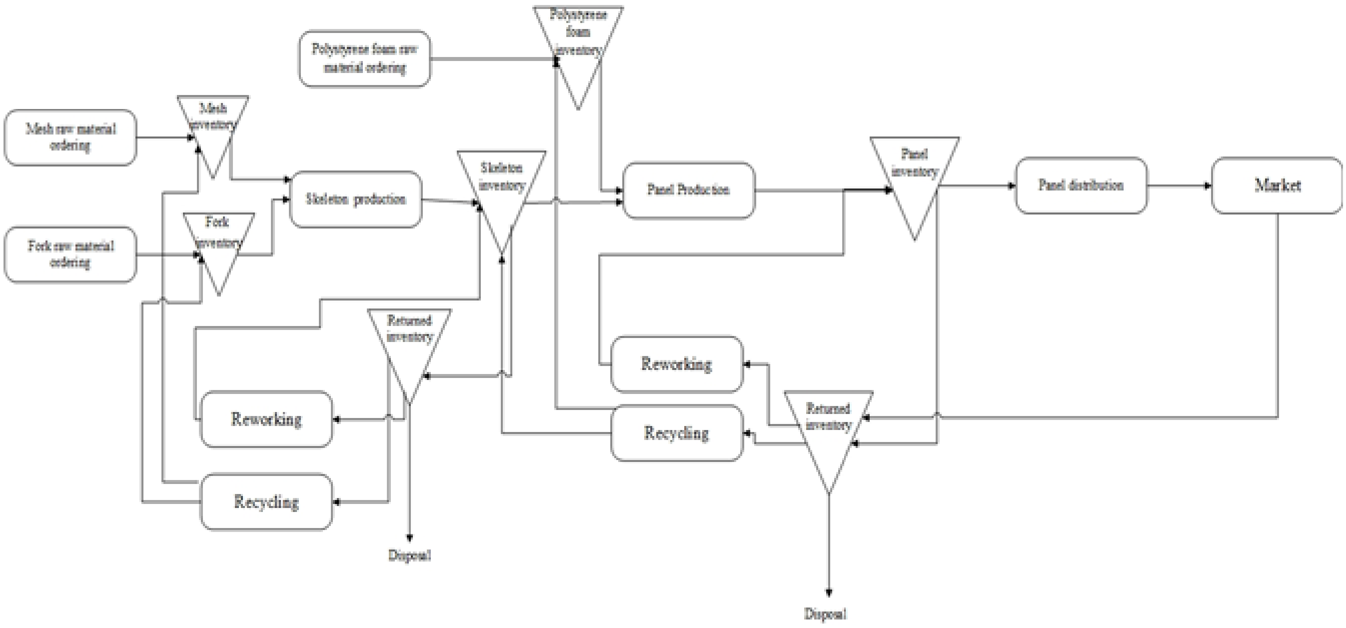

Figure 3. Production-inventory model of 3D panel

Each product often requires a specific process and production line. Mashhad Panel Barsava Manufacturing Company also has its own production process for its products. To produce 3D panels, first the raw materials for mesh ($s_{mb}(t)$) and fork ($s_{mc}(t)$) are ordered. The raw materials are then formed into a skeleton after welding. The amount of skeletal inventory is shown by $I_{mA_1}(t)$. At this stage, another raw material called polystyrene foam will be ordered to produce the 3D panel $s_{md}(t)$. The polystyrene foam ($s_{md}(t)$) and skeleton ($I_{mA_1}(t)$) inventory will be sent to ($P_{md}(t)$) and ($P_{mA_1}(t)$) in order to produce 3D panels, respectively. Some produced skeleton inventory may be refunded with ω_{m_1} rate. Among the returned skeletons, some will be recycled at the ω_{CA_1}rate and after separation, will be added to the mesh and fork inventory with $C_{rb}(t)$ and $C_{rc}(t)$, respectively. And some will be sent at the ω_{mA_1}rate for reworking. After combining polystyrene foam with the work in process in the previous stage, 3D panel will be produced. The production order of 3D panels is equal to ($s_{mA}(t)$) and from their inventory ($I_{mA}(t)$), some may be returned at the ω_{m_1} rate and the rest will be sent to the market as demand ($D(t)$) for distribution. From the returned 3D panel inventory ($I_{RA}(t)$), some will be sent at the ω_{RA} rate for reworking and will be added to the 3D panel inventory after reworking. Some will be recycled at the ω_{CA} rate that will be added to the polystyrene foam ($I_{md}(t)$) and skeleton ($I_{mA_1}(t)$) inventories. The rest will be disposed of at the ω_{dA} rate of perishable goods $\omega_{dA}(t)I_{RA}(t)$. Among the panels distributed in the market, the $R(t)$ is returned as much.

Two models will be introduced next. The lead time in each production stage will be considered as zero in the first model, and in the second model the lead time for each production stage will be considered. It is worth noting that the parameters used in both models are similar.

The values of the parameters and the beginning inventory amount related to Mashhad Panel Barsava Manufacturing Company are specified in Tables 2 and 3, respectively.

Table 2. Constant parameters

parametr	ω_{m1}	ω_{m2}	ω_{RA_1}	ω_{RA}	ω_{CA_1}	ω_{CA}	ω_{dA_1}	ω_{dA}	α	β	γ	δ	T	t	τ_{A_1}	τ_A	a	b
Value	0.05	0.03	0.8	0.7	0.15	0.2	0.05	0.1	2	20	1	1	6	1	0.1	0.05	0.01	0.1

Table 3. Inventory values at the beginning of the period

Variable	$I^0{}_{mb}$	$I^0{}_{mc}$	$I^0{}_{md}$	$I^0{}_{mA_1}$	$I^0{}_{mA}$	$I^0{}_{RA_1}$	$I^0{}_{RA}$
Value	100	100	20	20	5	3	4

4.1. First mode: Production-inventory optimal control model of 3D panels without considering the lead time

By placing the specified parameters in tables 2 and 3 in the category of equation (13), a dynamic system is obtained in which the coefficients of state and control variables form the entries of matrices A and B, which are defined as follows:

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & -0.05 & 0 & 0 & 0.8 & 0.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & -0.03 & 0 & 0.7 \\ 0 & 0 & 0.05 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0.03 & 0 & -1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{40} & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now consider the linear binomial optimal control (16). If the present penalty factors are defined as

$Q = diag\{15,16,10,14,3,8,9\}, K = diag\{120,190,140,110,145,130,125\}$ and $R = diag\{2,6,4,5,2,2,1,1,2\}$ matrices, the final answer is shown in Figures 4 and 5. Therefore, variables of state and control converge to their target values. Since the state and control functions are the deviation between the answers and their objective functions, according to

Figures 4 and 5, when $\Delta f\ (t) \rightarrow 0(f\ (t))$ it can be a state or control variable), it means that $f\ (t) - \hat{f}\ (t)$ is convergence to zero or $f\ (t) \rightarrow \hat{f}\ (t)$ that means functions (state or control) are convergent according to their purpose. Convergence to predetermined target values means that inventory variables and the level of orders and shipments have reached the desired values for managers to control, which will ultimately reduce costs.

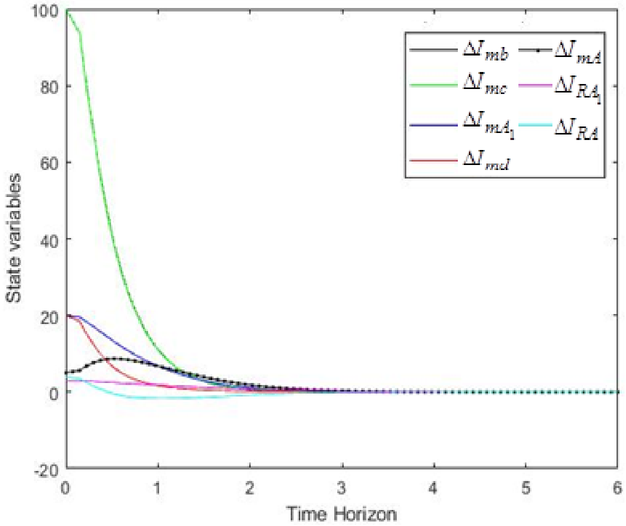

Figure 4. Final solution for state variables (without LTs)

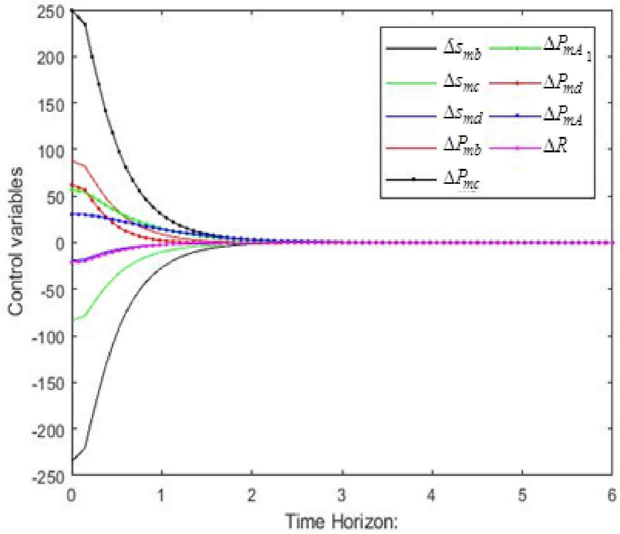

Figure 5. Final solution for control variables (without LTs)

4.2. Second mode: Production-inventory optimal control model of 3D panels with considering the lead time

Now, if the specified parameters in Tables 2 and 3 are included in the delayed optimal control model (22), the matrix of coefficients of the state and control variables $\acute{A}$ and $\acute{B}$ is obtained as

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0.04008 & 4.08 & -0.05 & 0 & 0 & -0.4012 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\ -0.001 & -0.81 & 0.0501 & 0.0497 & -0.03 & -0.0017 & 0.7 \\ 0 & 0 & 0.05 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0.03 & 0 & -1 \end{bmatrix} \quad B' = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -0.04 & -4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0.00001 & 0.001 & -0.0025 & 0 & 0 & 0.0025 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Penalty factors in the current linear binomial optimal control will be considered the same as the previous model, so are defined as $Q = diag\{15,16,10,14,3,8,9\}$ $K = diag\{120,190,140,110,145,130,125\}$ and $R = diag\{2,6,4,5,2,2,1,1,2\}$ matrices. The results of solving this model are shown in figures 6 and 7.

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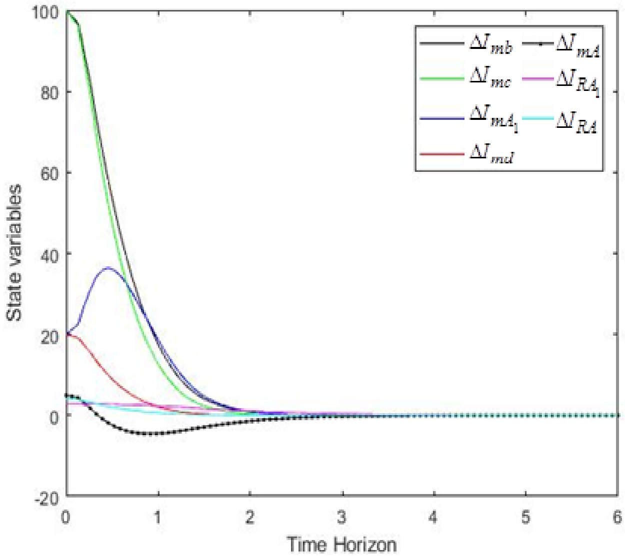

Figure 6. Final solution for state variables (with LTs)

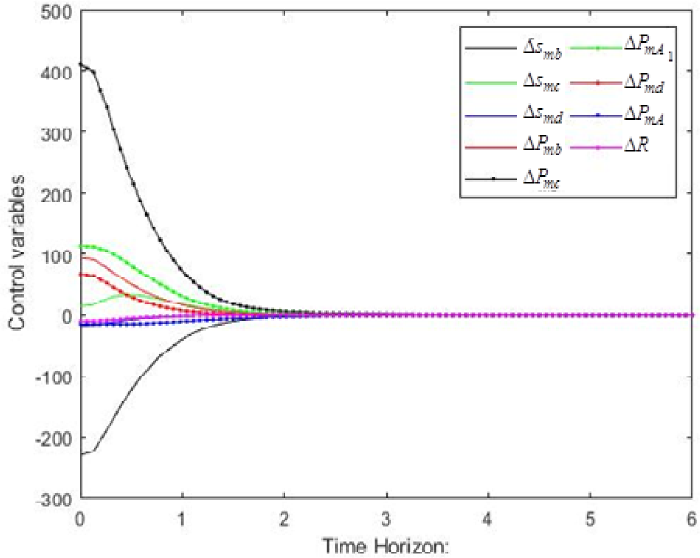

Figure 7. Final solution for control variables (with LTs)

Comparing the two figures 4 and 6, it can be clearly seen that the convergence velocity deviation of the inventories $\Delta I_{mA_1}, \Delta I_{RA_1}, \Delta I_{mA}, \Delta I_{RA}$ in the model without delay is greater than the delay mode, but the values of variables $\Delta I_{mb}, \Delta I_{mc}, \Delta I_{md}$ in both delayed and non-delayed modes are almost equal, because at the beginning of each step there is no delay for inventories related to raw materials.

Because there should be no additional fork and mesh raw materials in the skeleton production stage, the diagram related to the deviation from the desired level of the two variables of mesh and fork ($\Delta I_{mb}, \Delta I_{mc}$) is equal in the case of no delay. However, if the system is delayed, the convergence deviation rate from the desired level corresponding to the fork is more significant than that of the mesh. The consumption coefficient of the fork is greater than that of the mesh. By comparing the skeletal inventory, it can be seen that the deviation from the desired level is first increased and then converges to zero in the case of a delay system. However, the behavior of the diagram for the raw material of the foam is the same in both models.

Comparing the two figures 5 and 7, which are related to the deviation from the target level of control variables, the convergence rate is higher in the model without delay than the delay model.

Table 4. Comparison of objective function

Problem	Delay	Non-delay
The value of the objective J^*	351890	216878

5. *Sensitivity analysis for Q, K and R*

In this section, sensitivity analysis is performed in order to show the effect of changes in diagonal matrices Q, K and R and on the values of the objective function in both proposed models. The calculations and results that will be given in the continuation of this section are based on the information in tables 2 and 3.

5.1. *Changes in matrix Q entries*

As it is clear, by increasing the values q_i for $i = 1,2,,\dots,7$ its effect on the values of the objective function in the two proposed models have been investigated. The following 4 figures show the convergence status of the state and control variables for values $q_i = 3$ and $q_i = 11$ for $i = 1,2,,\dots,7$.

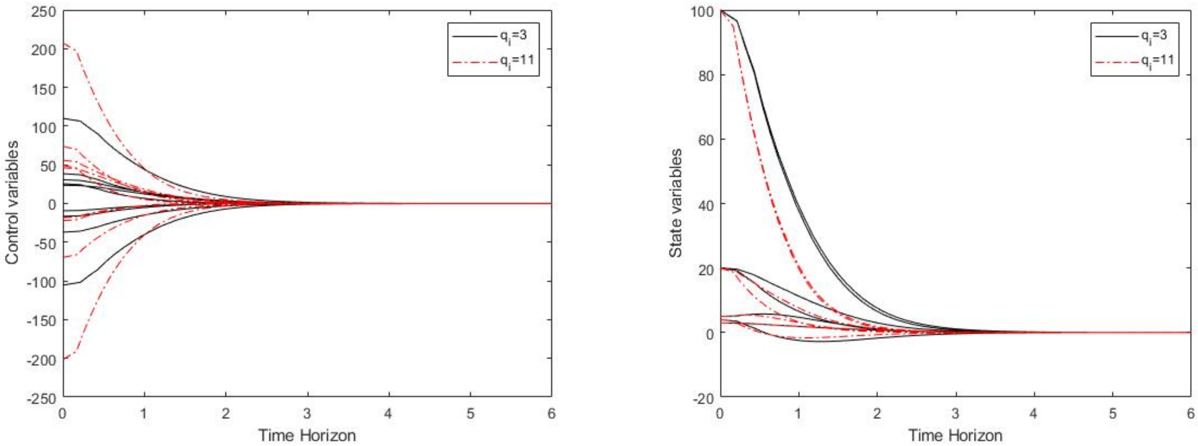

Figure 8. Comparing the convergence speed of state and control variables to their target for different values of q_i (Non Delay)

In general, it can be seen that as the matrix Q values increase, the state variables converge faster. As the matrix Q values increase, the penalty for deviating from the target values of the state variables increases. Because the objective function is a minimization type, $\Delta f(t) = f(t) - \hat{f}\,(t)$ converges faster to zero for the state variables, or by increasing the matrix Q values, the value of the objective function increases; However, it causes the convergence velocity of state variables to increase (in this example, zero), and convergence occurs faster. In all figures, the variables are grouped for two different values of the matrix.

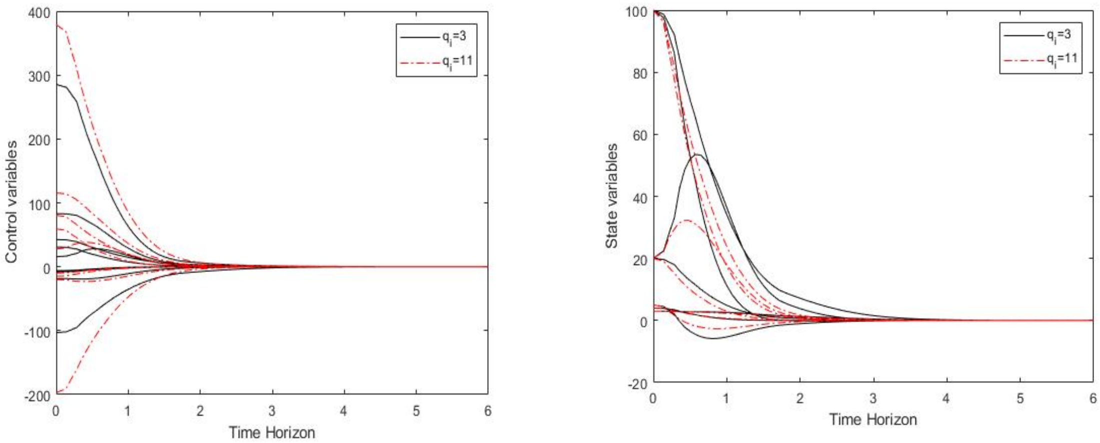

Figure 9. Comparing the convergence speed of state and control variables to their target for different values of q_i (Delay)

As shown in figures 8 and 9, as the values of the matrix increase, the convergence velocity of the control variables is supposed to be slower since the matrix Q is connected to the coefficients of the state variables.

5.2. Changes in matrix K entries

As it is clear, by increasing the values k_i for $i = 1,2, \dots ,7$, its effect on the values of the objective function in the two proposed models have been investigated. The following 4 figures show the convergence status of the state and control variables for values $k_i = 100$ and $k_i = 140$ for $i = 1,2, \dots ,7$.

According to the following 4 figures and the table above, the matrix K minimizes the final values of the state variables. Because the state variables' final values are very close to zero, the values of the table have changed slightly.

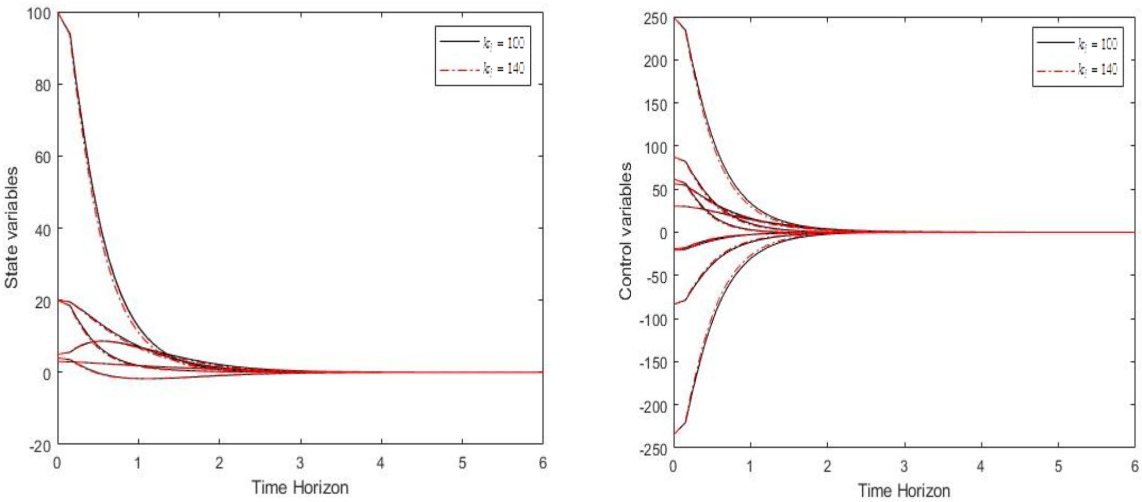

Figure 10. Comparing the convergence speed of state and control variables to their target for different values of k_i(Non-Delay)

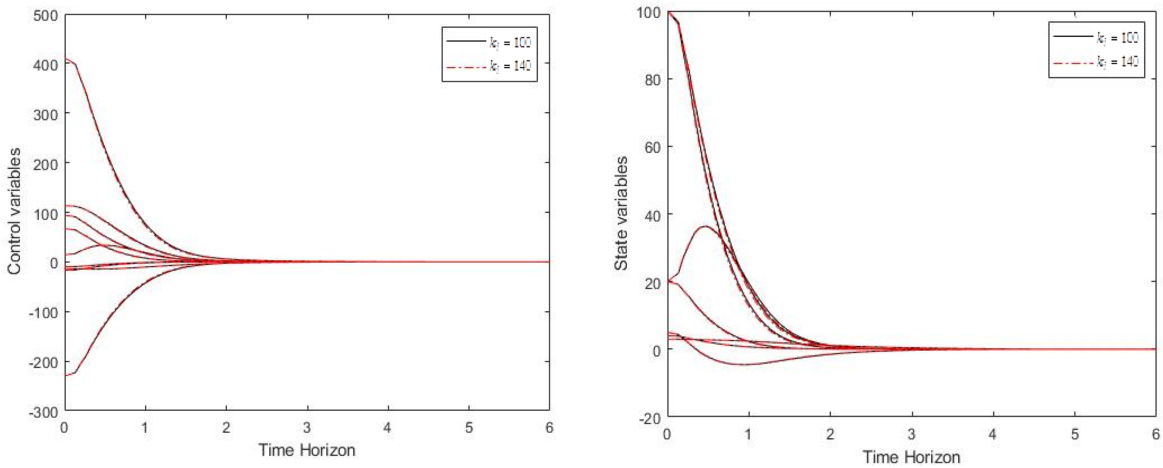

Figure 11. Comparing the convergence speed of state and control variables to their target for different values of k_i (Delay)

5.3. Changes in matrix R entries

As it is clear, by increasing the values r_i for $j = 1,2, \dots ,9$, its effect on the values of the objective function in the two proposed models have been investigated. The following 4 figures show the convergence status of the state and control variables for values $r_i = 2$ and $r_i = 6$ for $j = 1,2, \dots ,9$.

Increasing the matrix values as expected increases the objective function value. According to the following statistics, because the matrix is connected to the control variables' coefficients, the control variables converge more rapidly. As shown in Figure 13, since the coefficients of the state variables in the objective function have not changed, the convergence velocity of the state variables has slowed down.

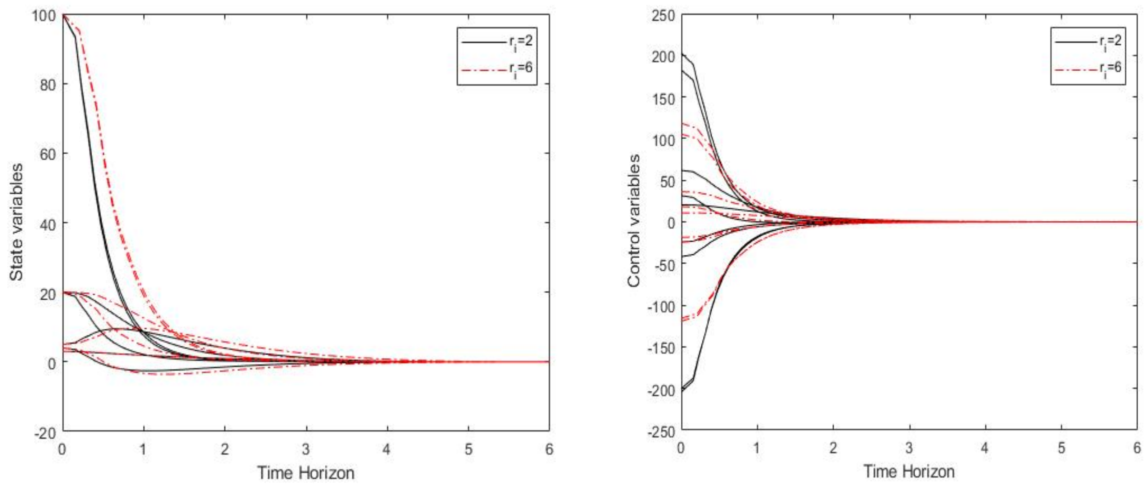

Figure 12. Comparing the convergence speed of state and control variables to their target for different values of r_i (Non-Delay)

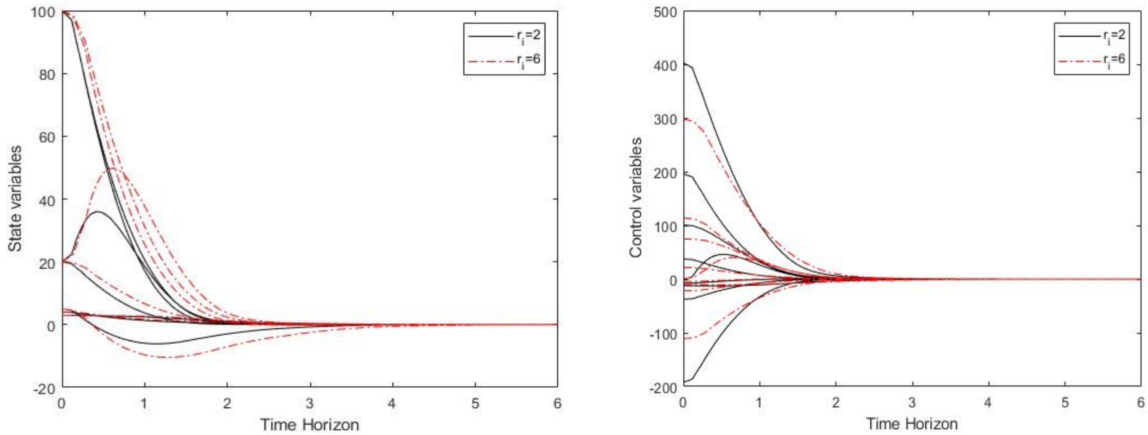

Figure 13. Comparing the convergence speed of state and control variables to their target for different values of r_i (Delay)

Therefore, it is possible to summarize the sensitivity analysis of changes in the values of the coefficients of state and control variables in the objective function as follows:

Table 5. Comparison of sensitivity analysis

Increase in the matrix values	State Variables	Control Variables
Q	Convergence is faster because Q is the coefficient of state variables	Convergence is slower because Q is the coefficient of state variables
R	Convergence is slower because R is the coefficient of control variables	Convergence is faster because R is the coefficient of control variables
K	Not very effective	Not very effective

As mentioned, the matrix Q form the coefficients of the state variables. As the values of the matrix Q entries increase, the state variables will converge faster. Since the matrix R form, the coefficients of the control variables, the control variables' values will converge more rapidly as the values of the matrix R increase. The matrix K is the final coefficients of state variables, and since the final values of state variables are close to zero, they will not have much effect on the values of the objective function.

Therefore, by increasing (in a specific range) the penalty coefficients, the convergence is accelerated. Results indicate that increasing the penalty coefficient in the objective function, accelerates the convergence of the state and control variables.

6. Discussion

As mentioned above, by studying previous research, it is observed that several optimal control models have been proposed to control production and inventory by MRP systems, which often did not take into account the lead time or include the return and reworking stages. In this study, a production-inventory optimal control model is proposed that eliminates this shortcoming.

Model implications are presented in two sections: theoretical implications, which mainly compare the proposed model with previous models, and practical implications, which include cases related to the interpretation of the results of the proposed model.

6.1. Theoretical implications

Based on Pooya and Pakdaman's (2019) proposed approach, an optimal continuous MRP control model can be modified to use computational aspects of CMRP. For example, state variables for inventory stores, control variables for the production, ordering, and demand programs, BOM system, and demand dependence on BOM can all indicate CMRP computational requirements. The current proposed model is a continuous MRP model that considers the delay due to production lead time (Pooya and Pakdaman, 2019) and a multi-stage production-inventory system (Pooya and Pakdaman, 2019). Since time is considered a continuous parameter, the lead time is considered in each production stage. Hence, the model is more realistic and feasible for all industries. It is also a proposed model for determining production values at any given time in discrete production processes, such as a workshop, handling the flow, and applying assembly lines.

One of the significant differences between CMRP and DMRP is LT. In the CMRP system, the LT value can be any decimal number, while in the DMRP system provided by Ignaciuk and Bartoszewicz (2010). The LT must be an integer; in this model, if the LT is a non-integer, it should be rounded to the next larger integer. However, the number of orders will be received with a delay. If the LT is rounded to the first smaller number, orders will be received sooner, which will create a surplus inventory and thus increase maintenance costs, and MRP objectives will not be met.

According to the model proposed by Foul and Tadj (2007), returned items will only be returned to the market for reworking. In this article's model, the produced items will be examined before distribution in the market to check the quality of the distributed items. The defective items percentage in the market also be reduced. For this purpose, the returned items should be identified in this stage. Additionally, the return stage for defective items is considered after distribution. Also, in the proposed research model, in addition to reconstruction, the recycling of items is considered. However, in Mishra's (2016) models, returned items will only be reworked, and the rest will be disposed of. In this case, the corruption amount in the items will be high, and their model's flexibility will be less than in real conditions.

The recycling stage of the returned items is a benefit of this model. Since recycled items can be used in the production process, manufacturers understand that product recycling can lead to increased profits. Returned items, in addition to manufactured goods, could also be work in process goods. Returned items can be work in process and manufactured items, some of which cannot be recycled and will be disposed of as spoiled items. Returned items may be reworked or recycled. It is assumed that the reproduction and recycling processes are complete, and all inputs have been converted into the desired output. Otherwise, if there is waste in the recycling and reproduction processes, the model will automatically adapt to new coefficients by adjusting its structure.

Compared to previous models, one significant benefit of this model is that the sum of each initial item is sufficient only for production in the next stage. It is a multiplier of the number and the amount of inventory used in the BOM. It prevents the development of surplus inventories.

Since the proposed optimal control model is linear-quadratic, an accurate answer could be obtained. However, approximate approaches can be used for large-scale problems. The state variables' positive values represent surplus inventory, which exceeds the warehouse capacity, and the negative values represent the warehouse capacity that has not been utilized fully. As shown in the figures, the inventories are close to the target values after short times when the order programs are set.

6.2. Practical implications

A three-level BOM model is presented in this study, which also offers the flexibility to produce products with higher BOM levels.

The finite capacity of activities at workstations leads to limitations such as production, reconstruction, and recycling goals. The inventory objective is to minimize the difference between the stock level and its precautionary reserve level. The purpose of disposing of items indicates the desired level of disposal of perishable items, and the capacity to order, release, and transport raw parts are among other goals. Also, in MPS, the target demand for the final product is usually defined as planned demand. Therefore, minimizing the difference between demand and demand target at any given time creates consistency in the production process with the planned demand.

Positive market values are due to the unnecessary release of goods relative to MPS' anticipated market. Similarly, negative values for demand indicate that product release is lower than the planned demand for MPS. However, as shown in the figures, the other variables are set so that the $\Delta P_{mA}(t)$ converged to zero. It means that demand converges to predetermined target values.

As shown in Figure 4, the changes in the 3D panel inventory have initially increased. Because the difference between the input variables of the sent skeleton levels and foam sent to this inventory has increased from the amount of 3D panel goods seen in figure 5, and then converges to zero. Because the amount of sending 3D panel items deviation from the desired level to the previous level has converged to zero, the difference between the amount of deviation of skeletal and foam items from the desired level of deviation of the final product from the desired level has increased. After the convergence of all three variables to zero, this difference is reduced and converges to zero. Therefore, the final product's increase is due to the over-shipment of two skeletal and foam items. Thus, to converge faster, the company should act as shown in Figure 4, without delay. In the state of delay, as shown in Figure 6, it sends two skeletal and foam goods.

According to Figure 4, the foam inventory rapidly converges to the desired amount. If Figure 2 is taken into account, the foam raw material's deviation level is removed from this inventory. The level of the deviation of the raw material shipment is included. Figure 5 shows that the convergence velocity of sent items is greater than the order variable's convergence velocity. This means that the input rate has decreased relative to the output, resulting in an increase in the variable convergence velocity of the foam inventory mode.

As the input and output variables to the skeleton's inventory deviate from its desired value, all of them are state variables that have been inputted and outputted at a constant rate, so no significant change in the values of this variable has occurred. It is uniformly converged to its target value, which is zero.

The amount of the state variable of the skeletal stock level has increased from the desired predetermined value overtime period [0, 0.6], if Figure 6 is taken into account. Because in this period, the convergence velocity of the control variables and the input state were higher than the convergence velocity of the output variables. However, after this period, the convergence velocity in the output variables has increased, resulting in a decrease in the deviation of the skeletal stock level from its desired value.

Analyzing the variable behavior of the final inventory mode of the product in Figure 6 shows that the inventory amount of this product first decreased and then converged to zero. According to Equation (20) and the coefficients of 3D panel product inventory variable in the matrix B' in "production -inventory Optimal control model of 3D panels considering production lead time" section, it can be seen the control variables of the fork, mesh, and skeleton order level are the input variables and the order level of the foam. The amount of sent 3D panel is the output ones. Due to the behavior of the above variables in figure 7, the negative value of the order levels of the two raw materials of mesh and fork and their coefficients in the matrix B' , the final product and the order level of the raw material of the foam are converged with delay to their target, which reduces the inventory variable value of the final product in the delayed state. The lack of 3D panel inventory in the mentioned range is due to the high level of ordering of the two raw materials of mesh, fork, and skeleton.

7. Conclusions and future suggestions

A quadratic linear optimal control model is modeled in this paper for a multi-stage production-inventory MRP system, which considers the lead time of production. A delayed dynamic system was deemed for this reason. In the proposed model, the value of the objective function indicates the convergence rate of the variables (state and control functions) to their respective values. The inventories are state variables, while the order variables, the number of shipment goods, and the amount of demand for work in process are control variables. Production lead time refers to the time required to set up and process machines, as well as the time required to transport items between different parts of the system. One of the innovations in the proposed optimal control model compared to previous models is that it provides functions, allowing the amount of items sent to the next stage to be as much as needed for production, thereby preventing surplus inventory.

In the delay model, the optimal delay control problem was approximated to a non-delay problem using the Taylor expansion. Finally, the exact answer to the problem without delay and the approximate answer to the delay problem were calculated and compared. A comparison of these responses revealed that the convergence of the non-delayed response to its target value is faster than that of the delayed one. Although the short lead time is assumed, the value of the delayed state's objective function is greater than the value of the non-delay problem. When the unit of time is large, the production lead time (less than one) can be reasonable. In the numerical simulation section, the time horizon $T = 6$ is considered, the lead time $\tau_A = 0.1$ means 2.4 hours. One of the advantages of the proposed model is that, unlike the existing methods, the production lead time can be continuous and a fraction of a unit of time.

Compared to the previous models, one of the significant advantages of this model is that the sum of each initial item until the next step is sufficient only for production. It is the quantity factor and quantity of inventory used in the BOM that prevent the development of excess inventory. In this paper, a general material requirement planning (MRP) system with deteriorating items for continuous production processes is considered, and an optimal control model is introduced, taking into account finite capacity constraints. It helps planners consider and model limited capacities during scheduling processes. In the optimal solution, when the state and control variables tend to zero, it means that they converge to the target values of the variables based on the optimal plan. According to Table 1, some existing optimal control models for inventory-production systems were multistage; however, the authors did not consider the quantity of ordered items (raw materials) in the final product. Also, they did not consider any lot-sizing policy. Therefore, our approach for CMRP is new, considers which it deems the BOM by determining the amount of ordered items in the finished product along with the continuous production approach in a continuous time framework, as well as the LFL lot-sizing policy.

The optimal control model, in which the time parameter is continuous, is capable of modeling the state and control variables so that they converge to their target values. Since this research focuses on three-level BOM products, it is recommended that further research be conducted for higher-level BOM products. Additionally, this research considers the Lot for Lot ordering method. In future research, other types of ordering systems could be considered. As a future study, restrictions on control and inventory variables could be considered. In this case, different optimization conditions must be obtained. Uncertainty methods could be used for the proposed model parameters, such as random planning. A closed-loop answer for the proposed system could also be obtained and analyzed. In this study, the values of control variables (order, send, and demand) and state variables (inventories) are calculated as time functions, so the answer is open-loop. That order, shipment, and demand could be calculated as inventory functions (closed-loop). The proposed algorithm could be developed to incorporate other variables of state and control if conditions change. However, when the number of production lead times increases or the number of stages increases, the approximation error also increases. It is because the first period of approximation was used in the development of the Taylor expansion. Other existing mathematical methods could be used to solve the system of delayed differential equations in both cases. Instead of linear relationships, nonlinear relationships between parameters could be used. In this case, the calculated error could also be increased.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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